On the Sandpile Groups of Circulant Graphs

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October 9, 2016
Outline

- Introduction to Sandpiles and Sandpile Groups
- Introduction to Circulant Graphs
- Previous Work and Motivation
- Our Problem and Results
- Future Work
Introduction to Sandpiles
A sandpile $c$ on a given undirected graph $\Gamma$ with a distinguished vertex, $s$, called the "sink", is a vector of non-negative integers indexed by the nonsink vertices.

Definition
What is a Sandpile?

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A sandpile is **stable** if and only if $c(v) < d_v$ for all $v \in V \setminus \{s\}$, where $c(v)$ is the integer associated to $v$ and $d_v$ is the degree of vertex $v$. If a sandpile is unstable then it sends 1 grain of sand along each of its adjacent edges. This process is called **toppling**. An unstable sandpile topples until it becomes stable.
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- If a sandpile is unstable then it sends 1 grain of sand along each of its adjacent edges. This process is called toppling.
- An unstable sandpile topples until it becomes stable.
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A certain collection of sandpiles on \( \Gamma \), called recurrences, form a group under stable addition.
The Sandpile Group

- Two sandpiles $c$ and $c'$ on $\Gamma$ can be added vertex-wise and then stabilized. This binary operation is called stable addition, denoted $\oplus$.

- A certain collection of sandpiles on $\Gamma$, called recurrences, form a group under stable addition.

- We call this group the sandpile group of $\Gamma$ and denote it $S(\Gamma)$. 
Definition

For a graph $\Gamma$ on $n$ vertices, the **Laplacian matrix**, $L(\Gamma)$, is an $n \times n$ matrix defined as $L(\Gamma) = D - A$, where $D$ is the diagonal matrix whose $(i, i)$ entry is the degree of $v_i$ and $A$ is the adjacency matrix of $\Gamma$.

$$L = \begin{pmatrix}
2 & -1 & -1 & 0 & 0 & 0 \\
-1 & 5 & -1 & -1 & -1 & -1 \\
-1 & -1 & 3 & -1 & 0 & 0 \\
0 & -1 & -1 & 2 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 \\
\end{pmatrix}$$
Sandpile Group cont.

**Definition**

The **reduced Laplacian**, $\tilde{L}(\Gamma)$, is obtained by deleting the row and column of $L(\Gamma)$ associated with the sink.

$$
\tilde{L} = \begin{pmatrix}
2 & -1 & -1 & 0 & 0 \\
-1 & 5 & -1 & -1 & -1 \\
-1 & -1 & 3 & -1 & 0 \\
0 & -1 & -1 & 2 & 0 \\
0 & -1 & 0 & 0 & 1
\end{pmatrix}
$$
The Fundamental Theorem of Sandpile Groups

Let $\Gamma$ be an undirected connected graph with a sink. Then, $S(\Gamma) \cong \mathbb{Z}^n / \tilde{L}\mathbb{Z}^n$.

Let $\text{diag}(k_1, k_2, \ldots, k_n)$ be the Smith normal form of $\tilde{L}$, where $\{k_1, k_2, \ldots, k_n\}$ are the invariant factors of $\tilde{L}$. Then,

$$S(\Gamma) \cong \mathbb{Z}_{k_1} \oplus \mathbb{Z}_{k_2} \oplus \cdots \oplus \mathbb{Z}_{k_n}.$$
Kirchhoff’s Matrix Tree Theorem

Theorem 2 (Kirchhoff’s Matrix Tree Theorem)

Let $\Gamma$ be an undirected connected graph and $L$ be its Laplacian matrix. Then

\[ \kappa(\Gamma) = \text{Det}(\tilde{L}) = |S(\Gamma)|, \]

where $\kappa(\Gamma)$ is the number of spanning trees on $\Gamma$. 
Introduction to Circulant Graphs
Definition

Let \( n \) be a positive integer and \( \{a_1, \ldots, a_m\} \) be a set of \( m \) integers such that \( 1 \leq a_j \leq n \) for each \( j \). We define the *circulant graph* \( C_n(a_1, \ldots, a_m) \) to be the graph with vertex set \( v_0, v_1, \ldots, v_{n-1} \) and edges connecting vertex \( v_i \) to each vertex \( v_{i+a_j} \), where addition is taken modulo \( n \) and \( \gcd(n, a_1, \ldots, a_m) = 1 \) to ensure the graph is connected.

Examples: \( C_8(1, 2), C_{13}(2, 5), C_{15}(1, 3, 4) \)
Disconnected Graphs

- Why do we care that $\gcd(n, a_1, \ldots, a_m) = 1$?
  - We want our graph to be a connected graph.

Example:

![Disconnected Graph Example](image)

Figure 1: $C_8(2, 4)$
A natural way of studying circulant graphs is through their adjacency matrices, which are also circulant. A $n \times n$ matrix $C$ is a **circulant matrix** if it has the following form:

$$C = \begin{pmatrix}
    c_0 & c_1 & c_2 & \cdots & c_{n-1} \\
    c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\
    c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    c_1 & c_2 & \cdots & c_{n-1} & c_0
\end{pmatrix}.$$
**Circulant Matrices**

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$
Previous Work and Motivation
In 2006, Hou, Woo, and Chen showed

\[ S(C_n(1, 2)) \cong \mathbb{Z}_{\gcd(n, F_n)} \oplus \mathbb{Z}_{F_n} \oplus \mathbb{Z}_{\frac{n F_n}{\gcd(n, F_n)}}, \]

where \( F_n \) is the Fibonacci sequence with roots \( F_0 = 1 \) and \( F_1 = 1 \).
Previous Work and Motivation

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where \( F_n \) is the Fibonacci sequence with roots \( F_0 = 1 \) and \( F_1 = 1 \).

- Example:

\[
S(C_6(1, 2)) \cong \mathbb{Z}_8^2 \oplus \mathbb{Z}_6, \quad F_6 = 8
\]

Figure 2: \( C_6(1, 2) \)
In 1986 Boesch and Prodinger proved that

$$|S(C_n(1, 2))| = nF_n^2$$
Previous work and motivation

- In 1986 Boesch and Prodinger proved that
  \[ |S(C_n(1, 2))| = nF_n^2 \]

- The number of spanning trees of $C_n(1, 3)$ has been proven (Yong, Acenjian 1996):
  \[ |S(C_n(1, 3))| = n(a_{n-2})^2, \text{ where} \]
  \[ a_n = \sqrt{2}(a_{n-1} + a_{n-3}) - a_{n-4}, \]
  \[ a_1 = 1, \ a_2 = 2\sqrt{2}, \ a_3 = 5, \ a_4 = 5\sqrt{2} \]
Problem
Is a nice structure (similar to the spanning tree computation) always seen in the sandpile group?

Answer: Our computations tell us that this is NOT true.
The Problem Statement

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- Recall: $|S(C_n(1, 2))| = nF_n^2$ and $S(C_n(1, 2)) \cong \mathbb{Z}_{\gcd(n,F_n)} \oplus \mathbb{Z}_{F_n} \oplus \mathbb{Z}_{nF_n/\gcd(n,F_n)}$. 

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- For $C_n(1, 3)$, we know $|S(C_n(1, 3))| = n(a_{n-2})^2$. 

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- For $C_n(1, 3)$, we know $|S(C_n(1, 3))| = n(a_{n-2})^2$.

- Question: Is it true that

  $$S(C_n(1, 3)) \cong \mathbb{Z}_{\gcd(n,a_n)} \oplus \mathbb{Z}_{a_n} \oplus \mathbb{Z}_{na_n/gcd(n,a_n)},$$

  for some $a_n$?

Answer: Our computations tell us that this is NOT true.
Is a nice structure (similar to the spanning tree computation) always seen in the sandpile group?

Recall: \( |S(C_n(1, 2))| = nF_n^2 \) and
\[
S(C_n(1, 2)) \cong \mathbb{Z}_{gcd(n,F_n)} \oplus \mathbb{Z}_{F_n} \oplus \mathbb{Z}_{nF_n/gcd(n,F_n)}.
\]

For \( C_n(1, 3) \), we know \( |S(C_n(1, 3))| = n(a_{n-2})^2 \).

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\]
for some \( a_n \)?

Answer: Our computations tell us that this is NOT true.
Consider $S(C_{21}(1, 3))$. Does it follow a similar structure as $S(C_n(1, 2))$?

$$S(C_n(1, 3)) = \mathbb{Z}_{\gcd(n, a_{n-2})} \oplus \mathbb{Z}_{a_{n-2}} \oplus \mathbb{Z}_{na_{n-2}/\gcd(n, a_{n-2})}$$

The sandpile group for $C_{21}(1, 3)$ is,

$$S(C_{21}(1, 3)) \cong \mathbb{Z}_{41} \oplus \mathbb{Z}_{41} \oplus \mathbb{Z}_{41 \times 13} \oplus \mathbb{Z}_{41 \times 13 \times 21}$$
Isomorphic Graphs
Equivalent Graphs

- Given $C_n(a, b)$ we do not need to look at all possible values for $a, b$.

$$C_n(a, b) \cong C_n(a^{-1}, b) \cong C_n(a, b^{-1}) \cong C_n(a^{-1}, b^{-1})$$

where $a^{-1}, b^{-1}$ are the respective inverses in $\mathbb{Z}_n$ under addition modulo $n$.

- Example:

\[ \text{Figure 3: } C_6(1, 2) \cong C_6(1, 4) \cong C_6(2, 5) \cong C_6(4, 5) \]
For $C_n(a, b)$ and $C_n(c, d)$, if we can construct $\varphi$ such that:

\[
\varphi(a) \mapsto c \quad \text{and} \quad \varphi(b) \mapsto d \quad \text{or} \quad d^{-1}
\]

or

\[
\varphi(a) \mapsto d \quad \text{and} \quad \varphi(b) \mapsto c \quad \text{or} \quad c^{-1}
\]

then we know that $C_n(a, b)$ is isomorphic to $C_n(c, d)$.
Let $\varphi(v) = 4v$. Then, $\varphi(1) = 4$, $\varphi(2) = 8 \mod(7) = 1$. That is, $C_7(1, 2) \mapsto C_7(1, 4)$.
The isomorphisms between circulant graphs allows us to focus on a smaller subset of graphs in order to understand the behavior of $S(C_n(a, b))$. 

<table>
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<th>Number of Graphs</th>
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<tr>
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<tr>
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<tr>
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<td>6</td>
</tr>
<tr>
<td>16</td>
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</tbody>
</table>
The Database of Circulant Graphs
Sage function to output isomorphic circulant graphs, invariant factors and the orders of the respective sandpile groups.

Transferring our data to MySQL in order to create a public database for any $C_n(a_1, \ldots a_m)$, $n < 30$. 
The Database

```
Tables_in_circulantgraphs

| n10 | n11 | n12 | n13 | n14 | n15 | n16 | n17 | n18 | n19 | n20 | n21 | n22 | n23 | n24 | n25 | n26 | n27 | n6  | n7  | n8  | n9  |
```

```
mysql> describe n8;

+----------------+---------+------+-----+----------------+---------+---+---------+---+---------+---------+---------+---+---------+---+---------+---------+---------+---+---------+---+---------+---------+---------+---+---------+---+---------+---------+---------+
| Field          | Type    | Null | Key | Default        | Extra   | n | graph_class       | n | inv_factors       | n | NULL    | NULL    | NULL    | n | NULL    | n | NULL    | NULL    | NULL    | n | NULL    | n | NULL    | NULL    | NULL    | n | NULL    | n | NULL    | NULL    | NULL    |
+----------------+---------+------+-----+----------------+---------+---+----------------+---+----------------+---+----------------+----------------+----------------+---+----------------+---+----------------+----------------+----------------+---+----------------+---+----------------+----------------+----------------+ ---+----------------+---+----------------+----------------+----------------+ ---+----------------+---+----------------+----------------+----------------+
```

```
``
ISOMORPHISMS FOR CIRCULANT GRAPHS WITH N=8

\[ C_8(1, 2) : C_8(2, 3) \]
\[ C_8(1, 3) : \text{None} \]
\[ C_8(1, 4) : C_8(3, 4) \]
\[ C_8(1, 2, 3) : \text{None} \]
\[ C_8(1, 2, 4) : C_8(2, 3, 4) \]
\[ C_8(1, 3, 4) : \text{None} \]
\[ C_8(1, 2, 3, 4) : \text{None} \]
NUMBER OF NON-ISOMORPHIC GRAPHS: 7

C_8(1, 2); Invariant Factors: [21, 168]
Invariant Factors Factored: [3 * 7, 2^3 * 3 * 7]
Order: 3528

C_8(1, 3); Invariant Factors: [4, 4, 4, 4, 16]
Invariant Factors Factored: [2^2, 2^2, 2^2, 2^2, 2^4]
Order: 4096

C_8(1, 4); Invariant Factors: [7, 56]
Invariant Factors Factored: [7, 2^3 * 7]
Order: 392

C_8(1, 2, 3); Invariant Factors: [3, 12, 48, 48]
Invariant Factors Factored: [3, 2^2 * 3, 2^4 * 3, 2^4 * 3]
Order: 82944

C_8(1, 2, 4); Invariant Factors: [51, 408]
Invariant Factors Factored: [3 * 17, 2^3 * 3 * 17]
Order: 20808

C_8(1, 3, 4); Invariant Factors: [3, 12, 12, 48]
Invariant Factors Factored: [3, 2^2 * 3, 2^2 * 3, 2^4 * 3]
Order: 20736

C_8(1, 2, 3, 4); Invariant Factors: [8, 8, 8, 8, 8, 8]
Invariant Factors Factored: [2^3, 2^3, 2^3, 2^3, 2^3]
Order: 262144
The Non-Isomorphic Graphs of $C_8(a_1, \ldots, a_m)$

Figure 6: $C_8(1, 2), C_8(1, 3), C_8(1, 4), C_8(1, 2, 3), C_8(1, 2, 3, 4), C_8(1, 2, 4)$
Observed Behavior

In small cases it had appeared that the following was true:

\[ C_n(k_1, k_2, \ldots, k_i) \cong C_n(j_1, j_2, \ldots, j_i) \iff S(C_n(k_1, k_2, \ldots, k_i)) \cong S(C_n(j_1, j_2, \ldots, j_i)). \]
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Our database confirms:

**Theorem**

*For* \( n \leq 19 \), \( C_n(k_1, k_2, ..., k_i) \cong C_n(j_1, j_2, ..., j_i) \iff S(C_n(k_1, k_2, ..., k_i)) \cong S(C_n(j_1, j_2, ..., j_i)). \)
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For \( n \leq 19 \),\( C_n(k_1, k_2, ..., k_i) \cong C_n(j_1, j_2, ..., j_i) \iff S(C_n(k_1, k_2, ..., k_i)) \cong S(C_n(j_1, j_2, ..., j_i)). \)

- However, this is not true for larger \( n \). Example:

\[
C_{20}(1, 2, 4, 9, 10) \not\cong C_{20}(1, 6, 8, 9, 10), \text{ but } \\
S(C_{20}(1, 2, 4, 9, 10)) \cong S(C_{20}(1, 6, 8, 9, 10)) \\
\cong \mathbb{Z}_{19} \oplus \mathbb{Z}_9^3 \oplus \mathbb{Z}_{28025} \oplus \mathbb{Z}_{224200}.
\]
A Bound on the Number of Invariant Factors
Theorem

For $S(C_n(1, 3))$ the maximum number of invariant factors is 5.

Proof: We can describe the sandpile group on a graph $\Gamma$ in the following manner:

$$\mathbb{Z} \oplus S(\Gamma) \cong \mathbb{Z}^n / \text{span}(\Delta_1, \ldots, \Delta_n),$$

where $\Delta_i$ is given by:

$$\Delta_i = d_i x_i - \sum_{v_j \text{ adjacent to } v_i} x_j$$

where $d_i$ the degree of $v_i$ and $x_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^n$, with 1 in the $i^{th}$ position. Note that $\Delta_i$ is the $i^{th}$ row of the Laplacian matrix.
For the case of $C_n(1,3)$, the following relation is true for $n \geq 7$:

$$\Delta_i = 4x_i - x_{i-1} - x_{i-3} - x_{i-n+1} - x_{i-n+3}$$

Now let $\bar{x}_i$ be the image of $x_i$ in $\mathbb{Z} \oplus S(C_n(1,3))$. Then we can solve for $\bar{x}_i$ to get the following relation:

$$\bar{x}_i = 4\bar{x}_{i-3} - \bar{x}_{i-2} - \bar{x}_{i-4} - \bar{x}_{i-6}.$$ 

For $i \geq 7$, $\bar{x}_i$ can be expressed as a linear combination of the values in $\{\bar{x}_1, \ldots, \bar{x}_j\}$, where $j < 7$. Note that $|\{\bar{x}_1, \ldots, \bar{x}_j\}| = 6$. Now, let vertex 6 be the sink of our graph. Then we have $\bar{x}_6 = 0$. Thus, for $S(C_n(1,3))$, there are at most $6 - 1 = 5$ generators.
In order to see the relation between the $\bar{x}_i$'s from the graph, we define the following object.

**Definition**

The **Claw** of $\bar{x}_i$ is the set of vertices adjacent to $v_{i-3}$.

![Diagram of the Claw](image)
The Claw for $\bar{x}_1$ in $C_8(1, 3)$

Figure 7: $\Delta_6 = 4\bar{x}_6 - \bar{x}_5 - \bar{x}_3 - \bar{x}_7 - \bar{x}_1$, gives $\bar{x}_1 = 4\bar{x}_6 - \bar{x}_5 - \bar{x}_3 - \bar{x}_7$
The $\overline{x}_i$'s for $C_8(1, 3)$

\[
\begin{align*}
\overline{x}_1 &= 4\overline{x}_6 - \overline{x}_5 - \overline{x}_3 - \overline{x}_7 \\
\overline{x}_2 &= 4\overline{x}_7 - \overline{x}_0 - \overline{x}_6 - \overline{x}_4 \\
\overline{x}_3 &= 4\overline{x}_0 - \overline{x}_1 - \overline{x}_7 - \overline{x}_5 \\
\overline{x}_4 &= 4\overline{x}_1 - \overline{x}_2 - \overline{x}_0 - \overline{x}_6 \\
\overline{x}_5 &= 4\overline{x}_2 - \overline{x}_1 - \overline{x}_7 - \overline{x}_3 \\
\overline{x}_6 &= 4\overline{x}_3 - \overline{x}_4 - \overline{x}_2 - \overline{x}_0 \\
\overline{x}_7 &= 4\overline{x}_4 - \overline{x}_5 - \overline{x}_3 - \overline{x}_1 \\
\overline{x}_8 &= 4\overline{x}_5 - \overline{x}_6 - \overline{x}_4 - \overline{x}_2
\end{align*}
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\overline{x}_5 &= 4\overline{x}_2 - \overline{x}_1 - \overline{x}_7 - \overline{x}_3 \\
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\bar{x}_4 &= 4\bar{x}_1 - \bar{x}_2 - \bar{x}_0 - \bar{x}_6 \\
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\end{align*}
A Bound on the Number of Invariant Factors for $S(C_n(a, b))$

**Theorem**

For $S(C_n(a, b))$ the maximum number of invariant factors is $2b - 1$.

**Proof:** We can generalize the result from the previous section for $S(C_n(a, b))$. Solving for $\overline{x}_i$, we get:

$$\overline{x}_i = 4\overline{x}_{i-b} - \overline{x}_{i-b+a} - \overline{x}_{i-b-a} - \overline{x}_{i-2b}.$$  

Again, note that determining $\overline{x}_i$ requires that we define $\overline{x}_{i-1}, \overline{x}_{i-2}, ..., \overline{x}_{i-2b}$. Let $\overline{x}_{i-2b}$ represent the sink vertex. Thus, for $S(C_n(a, b))$, we need at most $|\{\overline{x}_{i-1}, \overline{x}_{i-2}, ..., \overline{x}_{i-2b-1}\}| = 2b - 1$ generators.
Returning to $C_n(1, 3)$
Structure of the Invariant factors

- Can we classify for which $n$ the sandpile group, $S(C_n(1,3))$, has 2,3,4,5 invariant factors? For $n < 200$ we have,
Can we classify for which \( n \) the sandpile group, \( S(C_n(1,3)) \), has 2, 3, 4, 5 invariant factors? For \( n < 200 \) we have,

5 Invariant Factors
- Empty Lounge
- \( n = 8k \)
- \( n = 30k \)
- \( n = 175 \)

4 Invariant Factors
- \( n = 21, 28, 42, 63, \ldots \)
- \( n = 10k \)
- \( n = 25k \)
- \( n = 193 \)

3 Invariant Factors
- \( n = 12, 36, 108, 132, \ldots \)
- \( n = 44, 52, 68, \ldots \)
- \( n = 10k - 5 \)

2 Invariant Factors
- Empty Lounge
- Primes
- \( n \equiv 2 \mod 4 \)
The Sandpile Group of $C_n(1, 3)$

**Theorem**

*If $n$ is prime and the Smith normal form of the reduced Laplacian of $C_n(1, 3)$ produces 2 invariant factors, then we have*

$$S(C_n(1, 3)) \cong \mathbb{Z}_{a_n-2} \oplus \mathbb{Z}_{n \cdot a_n-2},$$

*where*

$$a_n = \sqrt{2}(a_{n-1} + a_{n-3}) - a_{n-4},$$

$$a_1 = 1, \ a_2 = 2\sqrt{2}, \ a_3 = 5, \ a_4 = 5\sqrt{2}.$$
**Proof**: Assume there are two invariant factors and $n$ is prime. By Yong and Acenjian’s work, we know

$$|S(C_n(1,3))| = n(a_{n-2})^2$$

Thus we have

$$k_1 \cdot k_2 = n \cdot a_{n-2}^2.$$

From the Smith normal form we know that $k_1$ divides $k_2$ and thus for some integer $c$,

$$c \cdot k_1^2 = n \cdot a_{n-2}^2.$$

Solving for $k_1$ gives

$$k_1 = \sqrt{\frac{n}{c}} \cdot a_{n-2}.$$
The Prime Proof

\[ k_1 = \sqrt{\frac{n}{c}} \cdot a_{n-2}. \]

Since \( n - 2 \) is odd, \( a_{n-2} \) will be an integer. Moreover, \( k_1 \) is also an integer. Since \( n \) is prime and \( c \in \mathbb{N} \), \( \sqrt{\frac{n}{c}} \) must either be irrational or 1 and because \( k_1 \) and \( a_{n-2} \) are integers it must be true that \( \sqrt{\frac{n}{c}} = 1 \), which gives us that \( k_1 = a_{n-2} \). Furthermore

\[ k_1 \cdot k_2 = n \cdot a^2_{n-2} \implies a_{n-2} \cdot k_2 = n \cdot a^2_{n-2} \implies k_2 = a_{n-2} \cdot n. \]

Thus, we conclude

\[ S(C_n(1, 3)) \cong \mathbb{Z}_{a_{n-2}} \oplus \mathbb{Z}_{n \cdot a_{n-2}}. \]
Future Work

- Continue expanding the database and create a website for everyone to access.

- Better understand $S(C_n(1, 3))$.

- Find an explicit formula for the sandpile group of $C_n(1, 3)$. 
Acknowledgements

We would like to thank Dr. Luis García Puente, Dr. Suzanne Weekes, Natalie Hobson and the MSRI IT Department.

This work was conducted during the 2016 Mathematical Sciences Research Institute Undergraduate Program (MSRI-UP) which is supported by the National Science Foundation (grant No. DMS-1156499) and the National Security Agency (grant No. H98230-116-1-0033).
References


