Varied Factor Ordering in 2-D Quantum Gravity and Sturm-Liouville Theory

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Brief introduction to the project

We are looking for the space of physically allowable wavefunctions for 2d quantum gravity

Solutions to Schrödinger’s equation

Solving for eigenfunctions of the Hamiltonian gives a basis for wavefunctions given the Hamiltonian is self adjoint

Sturm-Liouville Theory and orthogonality of eigenfunctions of the Hamiltonian

Conclusion
The Model

- Spacetime is $M \cong \mathbb{R} \times S^1$ equipped with a 2-metric $g$
- $x^0$ parametrizes time $\mathbb{R}$
- $\ell(t)$ is the arc length around the spatial manifold $\{x \in M \mid x^0 = t\}$
- Arc length is calculated with respect to the spatial part of the metric $g$
- Results in a Lagrangian for $\ell$ as a dynamical variable (Nakayama 1994)

$$L = \frac{1}{4l(x^0)} \left(i(x^0)\right)^2 - \lambda l(x^0) - \frac{(m + \frac{1}{2})^2}{l(x^0)}$$
Rationale for studying 2-D

- Our Hamiltonian for 2-D quantum gravity

\[ H_m = \Pi_\ell \ell \Pi_\ell + \left( m + \frac{1}{2} \right)^2 \ell^{-1} + \lambda \ell \]

- Notice the factor ordering ambiguity in the kinetic term

- Also our configuration variable \( \ell \) must be positive

- Both key issues in 4-D General Relativity that can be studied in our 2-D model
Varied Factor Ordering

- Our Hamiltonian

\[ H_m = \Pi_\ell \ell \Pi_\ell + \left( m + \frac{1}{2} \right)^2 \ell^{-1} + \lambda \ell \]

- Notice the factor ordering ambiguity

- Choose a two-parameter family of orderings where \( i + j + k = 1 \)

\[ \ell^i \Pi_\ell \ell^j \Pi_\ell \ell^k \]

- Apply Schrödinger quantization: \( \ell \rightarrow \hat{\ell}, \Pi_\ell \rightarrow -i\hbar \frac{d}{d\ell}, \hat{H}\psi = E\psi \)

\[
-\hbar^2 \frac{d^2 \psi}{d\ell^2} - \hbar^2 \left( 1 - (i - k) \right) \frac{d\psi}{d\ell} + \left( -\hbar^2 \left( \frac{(i - k)^2 - (i + k)^2}{4} \right) \ell^{-1} \right) + \left( m + \frac{1}{2} \right)^2 \ell^{-1} + \lambda \ell - E \right) \psi = 0.
\]
Self Adjointness

- When the Hamiltonian is of the form \( \hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \) it is formally self adjoint i.e.

\[
\langle \hat{H}\psi, \phi \rangle \equiv \int (\hat{H}\psi) \phi \, dx = \int \psi (\hat{H}\phi) \, dx \equiv \langle \psi, \hat{H}\phi \rangle
\]

- With varied factor ordering the operator is not self adjoint with respect to \( d\ell \) and the wave function is not normalizable.

- To impose formal self adjointness we use the measure

\[
\ell^{k-i} d\ell
\]
The equation we need to solve

\[-\hbar^2 \ell \frac{d^2 \psi}{d\ell^2} - \hbar^2 (1 - (i - k)) \frac{d \psi}{d\ell} + \left( -\hbar^2 \left( \frac{(i - k)^2 - (i + k)^2}{4} \ell^{-1} \right) + \left( m + \frac{1}{2} \right)^2 \ell^{-1} + \lambda \ell - E \right) \psi = 0 \]

The general form of a confluent hypergeometric equation is

\[ xy'' + (c - x)y' - ay = 0 \]

Substitutions are

\[ z = \alpha \ell \]

\[ \Psi(\ell) = e^{\beta z} z^\gamma y(z) \]

\[ \alpha, \beta, \gamma \text{ are constants that we choose, } a, c \text{ are to be determined.} \]
The 3 free parameters were

\[ \alpha = \frac{2\sqrt{\lambda}}{\hbar}, \quad \beta = -\frac{1}{2}, \]

\[ \gamma = \frac{(i - k) \pm \sqrt{(i + k)^2 + \frac{4(m + \frac{1}{2})^2}{\hbar^2}}}{2} \]

Which determines \( a \) and \( c \) to be

\[ a = \frac{-1 \pm \sqrt{(i + k)^2 + \frac{4(m + \frac{1}{2})^2}{\hbar^2}}}{2} + \frac{E}{2\hbar\sqrt{\lambda}} \]

\[ c = 1 \pm \sqrt{(i + k)^2 + \frac{4(m + \frac{1}{2})^2}{\hbar^2}} \]
Transformation

\[ zy''(z) + \left( \sqrt{(i + k)^2 + \frac{4(m + \frac{1}{2})^2}{h^2}} - z \right) y'(z) - \]

\[ \left( -1 \pm \sqrt{(i + k)^2 + \frac{4(m + \frac{1}{2})^2}{h^2}} \right) \left( \frac{-1 \pm \sqrt{(i + k)^2 + \frac{4(m + \frac{1}{2})^2}{h^2}}}{2} + \frac{E}{2\hbar\sqrt{\lambda}} \right) y(z) = 0 \]
Using the known solutions to the confluent hypergeometric equation we get the following energy eigenfunction and energy eigenvalues

\[ \Psi_n(\ell) = Ne^{-\sqrt{\lambda}/\hbar} \ell^\gamma L_n^{(c-1)}(\alpha \ell) \]

\[ E_n = 2\hbar \sqrt{\lambda} \left( n + \frac{1 \pm \sqrt{(i+k)^2 + \frac{4(m+\frac{1}{2})^2}{h^2}}}{2} \right) \]
Distinct Sets of Energy Eigenfunctions

- Notice that gamma has a plus-minus

\[ \gamma = \frac{(i - k) \pm \sqrt{(i + k)^2 + \frac{4(m + \frac{1}{2})^2}{\hbar^2}}}{2} \]

- The energy eigenfunction involves \( \gamma \)

\[ \Psi_n(\ell) = N e^{-\frac{\sqrt{\lambda}}{\hbar} \ell} \gamma L_n^{(c-1)}(\alpha \ell) \]

- Both the positive and negative branch yield a distinct set of energy eigenfunctions and eigenvalues

\[ \mu = \pm \sqrt{(i + k)^2 + \frac{4(m + \frac{1}{2})^2}{\hbar^2}} \]
Questions about the Eigenfunctions

- Are the eigenfunctions in $L^2$?

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$$\Psi_n(\ell) = Ne^{-\frac{\sqrt{\lambda}}{\hbar}} \ell^\gamma L_n^{(c-1)}(\alpha \ell)$$

- Always in $L^2$ when positive branch of $\mu$ is taken
- When negative branch of $\mu$ we need $\mu < 1$
Questions about the Eigenfunctions

- Are the eigenfunctions in $L^2$?

$$\Psi_n(\ell) = Ne^{-\frac{\sqrt{A}}{\hbar}} \ell^n L_{n}^{(c-1)}(\alpha \ell)$$

- Always in $L^2$ when positive branch of $\mu$ is taken
- When negative branch of $\mu$ we need $\mu < 1$

- Are the sets of eigenfunctions over complete in $L^2$?
Questions about the Eigenfunctions

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- Is each set of eigenfunctions mutually orthogonal?
Questions about the Eigenfunctions

- Are the eigenfunctions in $L^2$?

$$\Psi_n(\ell) = N e^{-\frac{\sqrt{\lambda \ell}}{\hbar}} \ell^\gamma L_n^{(c-1)}(\alpha \ell)$$

- Always in $L^2$ when positive branch of $\mu$ is taken
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- Are the sets of eigenfunctions over complete in $L^2$?

- Is each set of eigenfunctions mutually orthogonal?

- Are linear combinations of each set of eigenfunctions dense in $L^2$?
Sturm-Liouville Theory

Given a second order differential equation of the form

\[ P(x) y'' + Q(x) y' + R(x) y = 0 \]

we can put it in Sturm-Liouville form

\[ \left[ \mu(x) P(x) y'(x) \right]' + \mu(x) R(x) y(x) = 0 \]

If we began with an eigenvalue problem, the S.L. form gives us the S.L. eigenvalue problem

\[ p(x) y'(x)' + [\kappa r(x) - q(x)] y(x) = 0 \]

If \( y_1 \) and \( y_2 \) are eigenfunctions with distinct eigenvalues and appropriate boundary conditions then the solutions are orthogonal in \( L^2((a, b), r(x)dx) \)
Our Schrödinger’s equation with varied factor ordering is put into Sturm-Liouville form and we read off the eigenvalue problem

\[
\left[ \ell^{(1-(i-k))} \psi' \right]' + \left( \frac{E}{\hbar^2} \ell^{k-i} - \left( \frac{\lambda}{\hbar^2} \ell^{1-(i-k)} - \left( \frac{(i - k)^2 - (i + k)^2}{4} \ell^{-(1+i-k)} \right) \right) \right] \ell^{-(1+i-k)} = 0
\]

\[
q(\ell) = \frac{\lambda}{\hbar^2} \ell^{1-(i-k)} - \left( \frac{(i - k)^2 - (i + k)^2}{4} \right) \ell^{-(1+i-k)}
\]

\[
p(\ell) = \ell^{1-(i-k)} \quad r(\ell) = \ell^{k-i} \quad \kappa = \frac{E}{\hbar^2}
\]
Boundary Conditions

- The boundary condition

$$\lim_{b \to \infty} \left[ N_1 N_2 e^{\frac{-2\sqrt{\lambda \ell}}{\hbar}} P_1 P_2' \ell^{2\gamma+1-(i-k)} - N_1 N_2 e^{\frac{-2\sqrt{\lambda \ell}}{\hbar}} P_2 P_1' \ell^{2\gamma+1-(i-k)} \right]_{a}^{b} = 0$$

- Take positive branch of $\mu$

$$\lim_{a \to 0} \left[ N_1 N_2 e^{\frac{-2\sqrt{\lambda \ell}}{\hbar}} P_1 P_2' \ell^{1+\mu} - N_1 N_2 e^{\frac{-2\sqrt{\lambda \ell}}{\hbar}} P_2 P_1' \ell^{1+\mu} \right]_{a}$$

- Take negative branch of $\mu$

$$\lim_{a \to 0} \left[ N_1 N_2 e^{\frac{-2\sqrt{\lambda \ell}}{\hbar}} P_1 P_2' \ell^{1-\mu} - N_1 N_2 e^{\frac{-2\sqrt{\lambda \ell}}{\hbar}} P_2 P_1' \ell^{1-\mu} \right]_{a}$$
The eigenfunctions given by the positive branch of $\mu$ are orthonormal in $L^2((0, \infty), \ell^{k-i} d\ell)$.

The eigenfunctions given by the negative branch of $\mu$ are orthonormal in $L^2((0, \infty), \ell^{k-i} d\ell)$ when $|\mu| < 1$.

It is necessary to impose a boundary condition to get just one set of Sturm-Liouville eigenfunctions, which shows that a boundary condition is necessary information to characterize space of physically allowed wavefunctions.
Conclusion

- Applied varied factor ordering to the Hamiltonian

- Solved Schrödinger’s equation through a transformation to a confluent hypergeometric equation

- S.L. gives us proof of orthogonality of our eigenfunctions

- **Factor ordering matters**
Future work

- Try to find different transformations that will lead to different eigenfunctions

- Apply knowledge of factor ordering to 4-D General Relativity
Thank you for your time.
Questions?